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Bound on Scattering Elements for Irregularly Shaped Waveguides

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***Abstract*—A lower bound is obtained on the phase shifts that characterize the scattering of electromagnetic waves by obstacles in an irregularly shaped waveguide for which the form function is imprecisely known. A numerical example illustrates the method.**

I. Introduction

BOUNDS, variational principles, and variational-bound formulations for waveguide scattering given in the literature [1]–[4] are applicable only if the form function of the propagating mode is known precisely. Here a bound is presented on the scattering parameters (phase shifts) for an irregularly shaped waveguide where the form function is not known analytically. The bound is not stationary, in that the error (the difference between the bound and the true value) is of the first order. A dielectric right circular cylinder placed parallel to the axis of propagation in a rectangular waveguide serves as an illustration of the technique. Relatively close lower bounds on the phase shifts are obtained with a simple trial function used in place of the known form function of the TE_{10} mode.

Variational-bound scattering principles, in which the error is of the second order (stationary) and of known sign, have recently been formulated for quantum mechanical problems containing imprecise information about the target ground-state function [5]. Such formulations can most probably be generalized and extended to irregular waveguides since the form function of the propagating mode is the analog of the ground-state function. However, the variational-bound principles would entail the evaluation of far more difficult expressions than appear in the bound considered here.

II. Theory

In this section, we derive a lower bound on the phase shift of an electromagnetic wave scattered by an obstacle in an arbitrarily shaped waveguide for which precise information about the form function of the propagating mode $\vec{e}(x,y)$ is lacking. We define an operator P such that for the electric field intensity $\vec{E}(x,y,z)$

$$P\vec{E}(x,y,z)=\vec{e}(x,y)\int \vec{e}(x',y')\cdot\vec{E}(x',y',z)\,d\sigma' \qquad (1)$$

where $d\sigma' = dx'\,dy'$ and where $\vec{e}$ is normalized. P picks out by a Fourier coefficient calculation the propagating mode of $\vec{E}$; similarly, the operator

$$Q \equiv 1-P \qquad (2)$$

projects out the evanescent modes of $\vec{E}$. It follows from Maxwell's wave equation

$$-\nabla\times(\nabla\times\vec{E})+\epsilon(\omega/c)^2\vec{E}=0 \qquad (3)$$

that [4]

$$P\left[H+HQ\frac{1}{Q(W-H)Q}QH-W\right]P\vec{E}=0 \qquad (4)$$

where ϵ, ω, and c are the relative permittivity, the angular frequency, and the speed of light, respectively. Also

$$\begin{aligned} H&=T+V\\ T&=-\nabla^2\\ V&=\nabla\nabla\cdot+(1-\epsilon)(\omega/c)^2\\ W&=(\omega/c)^2. \end{aligned} \qquad (5)$$

Let η be the phase shift determined from (4) and η^P the phase shift associated with the so-called static equation

$$P(H-W)P\vec{E}^P=0 \qquad (6)$$

where

$$P\vec{E}^P=\vec{e}(x,y)f_P(z).$$

It is shown [4] that η^P represents a lower bound on the exact phase shift

$$\eta \geqslant \eta^P. \qquad (7)$$

Equation (6) can be written

$$\left[-d^2/dz+\omega_c^2+\bar{V}-W\right]f_P(z)=0 \qquad (8)$$

where ω_c^2 is the square of the cutoff frequency of the propagating mode

$$\omega_c^2=\int \vec{e}\cdot\left[-(\partial^2/\partial x^2)-(\partial^2/\partial y^2)\right]\vec{e}\,d\sigma \qquad (9)$$

and where the average potential

$$\bar{V}=\int \vec{e}\cdot V(x,y,z)\vec{e}\,d\sigma \qquad (10)$$

represents the effect of the obstacle weighted with the square of the mode function over the guide's cross section. Equation (8) cannot be written down explicitly for an irregular waveguide since $\vec{e}$ is not known exactly.

Now, an expression $B(z)$ can be constructed such that $B-\bar{V}$ is at least of the first order and the condition

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$$B \geqslant \bar{V} \tag{11}$$

is satisfied. Also, it follows from the Rayleigh–Ritz principle that

$$\omega_{ct}^2 - \omega_c^2 \geqslant 0 \tag{12}$$

(and stationary), where

$$\omega_{ct}^2 = \int \vec{e}_t \cdot \left[-(\partial^2/\partial x^2) - (\partial^2/\partial y^2) \right] \vec{e}_t \, d\sigma \tag{13}$$

is the square of the trial cutoff frequency of the propagating mode, and $\vec{e}_t(x,y)$ is a normalized trial form function. It follows then from the monotonicity theorem [4] that the phase shift η^B obtained from the differential equation

$$\left[-d^2/dz^2 + \omega_{ct}^2 + B - W \right] f_B(z) = 0 \tag{14}$$

satisfies the relation

$$\eta \geqslant \eta^P \geqslant \eta^B. \tag{15}$$

Thus η^B represents a first-order lower bound on the true phase shift η.

From the Gramm determinant inequality and the Schwartz inequality, Weinhold [6]–[7] obtains a lower bound on the expression $\int \psi^* A \psi \, d\tau$, where ψ is a normalized wave function, A is a positive definite operator, and $d\tau$ is a volume element. Since V is a negative definite operator (we do not consider plasma obstacles for which $1 > \epsilon$), a lower bound $-B$ on $-\bar{V}$ patterned after that of Weinhold is

$$\int \vec{e} \cdot (-V) \vec{e} \, d\sigma \geqslant \frac{\left\{ S \int \vec{e}_t \cdot (-V) \vec{e}_t \, d\sigma - (1 - S^2)^{1/2} \left[\int \vec{e}_t \cdot V^2 \vec{e}_t \, d\sigma - \left(\int \vec{e}_t \cdot V \vec{e}_t \, d\sigma \right)^2 \right]^{1/2} \right\}^2}{\int \vec{e}_t \cdot (-V) \vec{e}_t \, d\sigma} \equiv -B \tag{16}$$

where

$$S = \int \vec{e} \cdot \vec{e}_t \, d\sigma. \tag{17}$$

The approach of the so-called overlap integral S toward unity is the criterion of accuracy of the trial function $\vec{e}_t$. Although S is not known, the inequality (16) is valid when S is replaced by a lower bound, the Eckart bound, on its true value. A lower bound on S is given by

$$S^2 \geqslant (\omega_{1c}^2 - \omega_{ct}^2)/(\omega_{1c}^2 - \omega_c^2) \tag{18}$$

where ω_{1c} is the cutoff frequency of the first evanescent mode. The cutoff frequencies ω_c and ω_{1c} can be determined experimentally. In case their experimental values are not known, the expression (18) is sustained if ω_c and ω_{1c} are replaced by lower bounds to their true values.

III. Numerical Example

For irregular waveguides, the Rayleigh–Ritz principle and other methods [8]–[9] serve as a tool for the determination of the variational parameters contained in the trial function.

A numerical example is presented by a dielectric right circular cylinder located in a rectangular waveguide parallel to the z axis of propagation, as shown in Fig. 1.

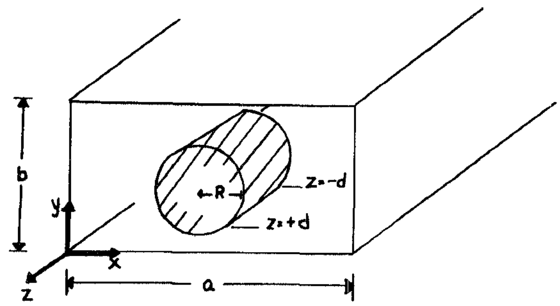

Fig. 1. A dielectric right circular cylinder in a rectangular waveguide.

The following parameters have been chosen:

$$(\omega/c)^2 = 2(\pi/a)^2$$
$$(b/a) = 1/2$$
$$\epsilon = 2 \tag{19}$$

where a and b are the wide and narrow dimensions of the waveguide. The cylinder with radius $R = a/4$ and with axis centered at $a/2$ and $b/2$ extends from $z = -d$ to $z = d$, where $d = a/2$. This simple (artificial) example, whose form function $\vec{e} = \vec{j}(2/ab)^{1/2} \sin(\pi x/a)$ is of course known, has been chosen in order to compare the quality of the bound with known results. We use the normalized trial function $\vec{e}_t$ instead of $\vec{e}$, where

$$\vec{e}_t = \vec{j}\,(30/ab)^{1/2} x(a-x)/a^2 \tag{20}$$

and where $\vec{j}$ is a unit vector in the y direction. Although $\vec{e}_t$ does not contain any variational parameters, it is a good approximation to $\vec{e}$. From (13)

$$\omega_{ct}^2 = 10/a^2 \tag{21}$$

while ω_c^2 is, from (9), $(\pi/a)^2$. With $\omega_{1c}^2 = (2\pi/a)^2$, it follows from (18) that

$$S^2 \geqslant 0.9956. \tag{22}$$

One obtains from (16) and (18)

$$B = -1.294(\pi/a)^2. \tag{23}$$

By the way, we can compare the bound B with the exact value of $\bar{V}$. From (10)

$$\bar{V} = -2\left(\frac{\pi}{a}\right)^2 \frac{\pi R^2}{ab} \left[1 + \frac{J_1(2\pi R/a)}{(\pi R/a)} \right]$$
$$= -1.3615(\pi/a)^2$$

where J_1 is a Bessel function and $R = a/4$.

The even η_e^B and odd η_o^B phase shifts corresponding to (14) are

$$\eta_o^B = 13°34'$$

$$\eta_e^B = 30°00'. \tag{24}$$

The upper and lower bounds on the phase shifts obtained by the quadratic Kato method [10] with the exact $\vec{e}$ are

$$14°30' \leqslant \eta_o \leqslant 15°30'$$

$$31°8' \leqslant \eta_e \leqslant 32°18'.$$

IV. Conclusion

A formulation has been presented for determining lower bounds on the phase shifts of dielectric obstacles in irregular waveguides. Relatively close lower bounds have been obtained with a simple trial function. This is to be expected for this type of obstacle for the permittivity value considered because the higher order evanescent modes contribute little excess phase shift. Of course, the bound may be improved by going to a better trial function with variational parameters. The extension to multimode waveguides is immediate.

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Propagation of Magnetostatic Waves Along Curved Ferrite Surfaces

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Abstract—**Electromagnetic equations have been appropriately transformed and solved in order to investigate the propagation of magnetostatic waves in curved geometries. The results have been utilized to study magnetostatic propagation along the surface of a cylindrically curved slab of ferrite in the azimuthal direction. In the case of an unbacked- or a metal-backed slab, it is found that the effect of curvature is to slightly reduce the phase as well as the group velocity by a constant factor throughout the frequency range of allowed modes. However, under favorable conditions, the presence of a dielectric layer between ferrite and metal leads to a strong enhancement in the propagation constant. It is also found that an axially magnetized homogeneous ferrite cylinder cannot support magnetostatic surface waves propagating along its curved surface in the azimuthal direction.**

I. Introduction

MAGNETOSTATIC wave propagation along curved ferrite surfaces is an area of importance on account of its relevance to a variety of practical situations, e.g., magnetostatic surface wave resonant modes of a ferrite slab with rounded edges [1], magnetic surface wave ring interferometer [2] characterized by propagation along the curved surface of a dielectric cylinder with ferrite sleeve, projected magnetostatic waveguide bends [3], scattering of electromagnetic waves by composite ferrite cylinders, etc. While the effect of curvature on guided wave propagation in hollow metallic [4], [5] and dielectric [6], [7] structures has been investigated in the past, similar studies in the case of ferrites are not available. In this paper, we have investigated the effect of curvature on propagation characteristics of magnetostatic surface waves in ferrites magnetized transverse to the direction of propagation. In Section II, the electromagnetic equations have been appropriately transformed and solved for the curved geometry. In Section III, the dispersion relation has been obtained in the general case of magnetostatic wave propagation, in azimuthal direction, and in a cylindrically curved

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